FINAL EXAM (SUPPLEMENTARY) ISI BANGALORE JUNE 2017

B MATH ALGEBRA II 100 POINTS

Notes.
(a) Justify all your steps.
(b) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.
(c) By default, F denotes a field.

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1. [30 points] Let A denote the matrix

$$A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$$

  (i) Find the characteristic polynomial, the eigenvalues and a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.
  (ii) Find an invertible 3×3 matrix X such that XAX^{-1} is a diagonal matrix and verify by explicitly multiplying out that XAX^{-1} is indeed diagonal.
  (iii) Using (ii) or otherwise calculate A^{100}.
  (iv) Find the minimal polynomial of A.

2. [16 points] Let $v_1, \ldots, v_r$ be an orthonormal set of vectors in $\mathbb{R}^n$.
  (i) Prove that there exists a matrix $A \in O_n(\mathbb{R})$ such that the i-th column of A is v_i.
  (ii) If $w_1, \ldots, w_r$ is another orthonormal set of vectors in $\mathbb{R}^n$, prove that there is a matrix $B \in O_n(\mathbb{R})$ such that $Bw_i = v_i$.

3. [16 points] Let $T\colon V \to W$ be a linear map where V, W are finite dimensional vector spaces.
  (i) Define $\mathrm{rank}(T)$ and $\mathrm{Nullity}(T)$.
  (ii) Prove that $\mathrm{rank}(T) + \mathrm{Nullity}(T) = \dim(V)$.

4. [12 points] Prove or Disprove:
For all possible choices for every entry $*$, the two matrices A and B are similar.

$$A = \begin{pmatrix} 1 & * & * & * \\ 0 & 2 & * & * \\ 0 & 0 & 3 & * \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 0 & 0 & 0 \\ * & 3 & 0 & 0 \\ * & * & 2 & 0 \\ * & * & * & 1 \end{pmatrix}$$

5. [16 points]
  (i) Let A be a normal matrix. Prove that if $Av = \lambda v$ then $A^*v = \bar{\lambda}v$.
  (ii) Let A be a Hermitian symmetric matrix. Prove that every eigenvalue of A is real.

6. [10 points] Prove that every element of $SO_2(\mathbb{R})$ corresponds to a rotation in $\mathbb{R}^2$.